Sydney Girls High School

2021 Alternate Task 4

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using a black pen
- Calculators approved by NESA may be used
- A NESA reference sheet has been provided for use
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks : 40

Section 1 – 10 marks (pages 2 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 30 marks (pages 7 – 12)

- Attempt Questions **11 – 13**
- Allow about 45 minutes for this section

THIS IS NOT A TRIAL PAPER

It does not reflect the format or the content of the 2021 HSC Examination paper in this subject.

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

For each question, select the correct response A, B, C or D.

List the correct response only on YOUR writing paper for questions 1 – 10.

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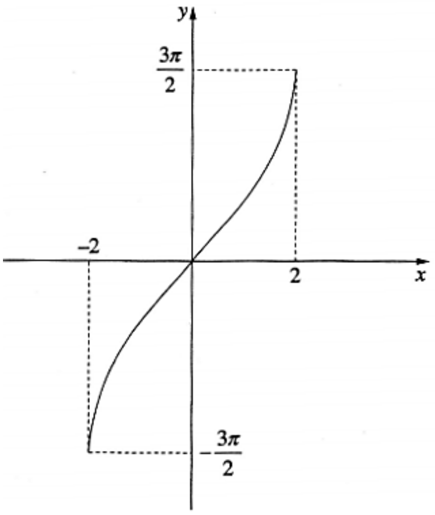
1. Which function best describes the following graph?

(A) $y = 3\sin^{-1} 2x$

(B) $y = \frac{3}{2}\sin^{-1} 2x$

(C) $y = 3\sin^{-1}\frac{x}{2}$

(D) $y = \frac{3}{2}\sin^{-1}\frac{x}{2}$



2. In the cartesian plane, a vector perpendicular to the line $3x + 2y + 1 = 0$ is

(A) $3\underset{\sim}{i} + 2\underset{\sim}{j}$

(B) $-\frac{1}{2}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j}$

(C) $2\underset{\sim}{i} - 3\underset{\sim}{j}$

(D) $\frac{1}{2}\underset{\sim}{i} - \frac{1}{3}\underset{\sim}{j}$

3. Which group of three numbers could be the roots of the polynomial equation

$$x^3 + px^2 - 26x + 24 = 0\ ?$$

(A) $2, 3, 4$

(B) $1, -6, 4$

(C) $-1, -2, 12$

(D) $-1, -3, -8$

4. $$\frac{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}}{\cot\frac{\theta}{2} + \tan\frac{\theta}{2}} =$$

(A) $\cos\theta$

(B) $\sec\theta$

(C) $\tan\theta$

(D) $\cot\theta$

5. Which expression is equal to $\int \cos^2 \frac{2x}{5}\, dx$?

(A) $\frac{x}{2} - \frac{2}{5}\sin\frac{4x}{5} + C$

(B) $\frac{x}{2} + \frac{2}{5}\sin\frac{4x}{5} + C$

(C) $\frac{x}{2} - \frac{5}{8}\sin\frac{4x}{5} + C$

(D) $\frac{x}{2} + \frac{5}{8}\sin\frac{4x}{5} + C$

6. In the expression $(2x + k)^6$ the coefficients of x and x^2 are equal.
What is the value of k?

(A) 5

(B) 6

(C) 11

(D) 12

7. A group consisting of two adults, two boys and two girls is to be seated at a round table. The adults are to be seated together. The boys and girls are to sit in alternating positions? How many different seating arrangements are possible?

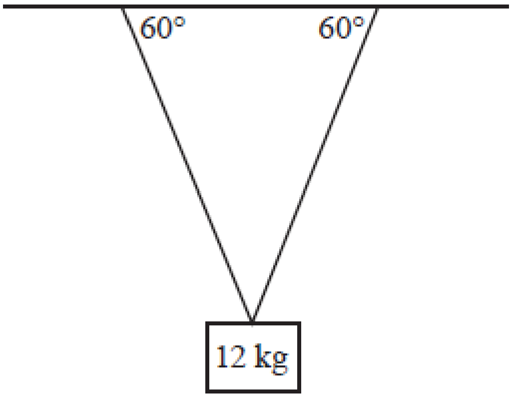
(A) 8

(B) 16

(C) 24

(D) 30

8. A 12kg mass is suspended in equilibrium from a horizontal ceiling by two identical light strings. Each string makes an angle of 60° with the ceiling, as shown. If the force of gravity is g m/s^2 then the magnitude, in newtons, of the tension in each string is equal to:



(A) $12g$

(B) $24g$

(C) $4\sqrt{3}g$

(D) $8\sqrt{3}g$

9. The volume V of a spherical balloon of radius r mm is increasing at a constant rate of 800 mm^3 per second. What is the rate of change of the radius with respect to time?

(A) $\dfrac{\pi r}{100}$ mm/second

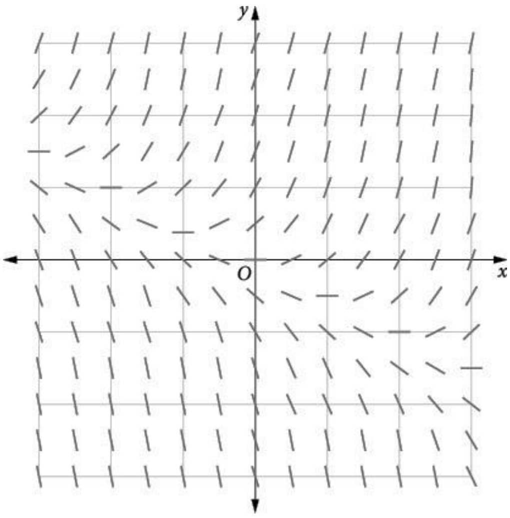
(B) $\dfrac{100}{\pi r}$ mm/second

(C) $\dfrac{\pi r^2}{200}$ mm/second

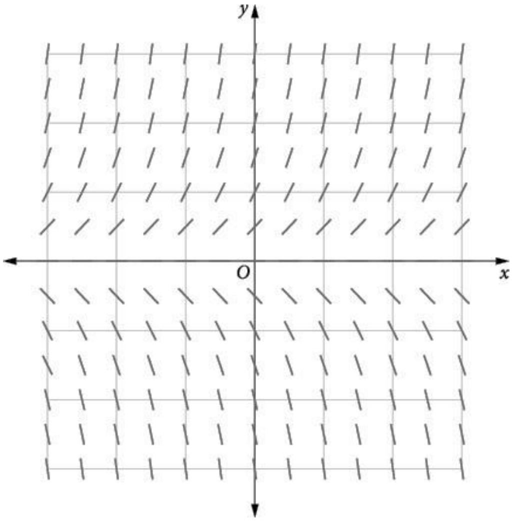
(D) $\dfrac{200}{\pi r^2}$ mm/second

10. Which of the following direction fields could have the differential equation $\dfrac{dy}{dx} = x - ky$ as a solution if $k > 0$?

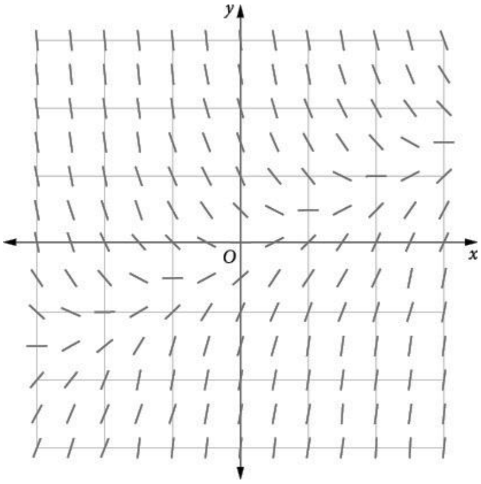
(A)



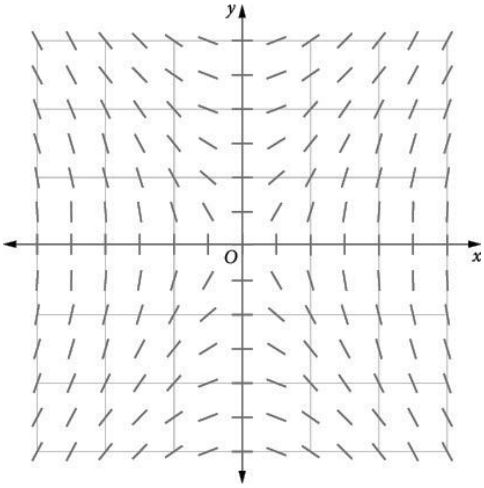
(B)



(C)



(D)



Section II

30 marks

Attempt Questions 11 – 13
Allow about 45 minutes for this section

Begin a new page for each question.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 **Begin a new Page** **(10 marks)**

(a)

(i) Show that $(\boldsymbol{x}-3)$ is a factor of the polynomial **1**

$P(x)=x^3-4x^2+x+6$.

(ii) Hence, express $P(x)$ in factored form. **2**

(b) Evaluate $\lim_{x\to 0}\left(\dfrac{\sin\dfrac{x}{3}}{4x}\right)$ **1**

(c) Solve: $\dfrac{x^2-9}{3x}>0$ **2**

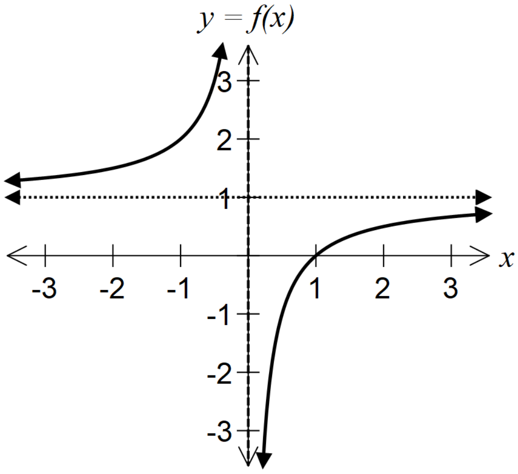
(d) Evaluate: **2**

$$\int \frac{2x}{\sqrt{25-x^2}}\,dx$$

Question 11 continues on the next page.

Question 11 (continued)

(e) Consider the function $f(x) = \dfrac{x-1}{x}$, which is graphed below:



Using the graph of $y = f(x)$, sketch the function $y = \{f(x)\}^2$. **2**

End of Question 11

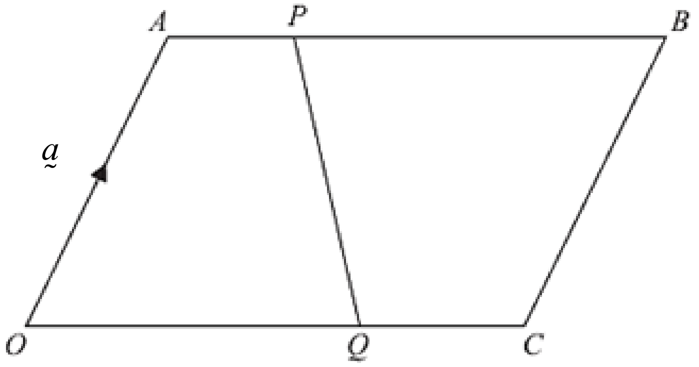
Question 12 Begin a new page. **(10 marks)**

(a) In the diagram below, $OABC$ is a parallelogram. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$ **3**

P is the point on AB such that $AP = \frac{1}{4}AB$.

Q is the point on OC such that $OQ = \frac{2}{3}OC$.

Find $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$, giving your answer in simplest form.



(b) Find the volume, in terms of π, of the solid formed when the area **3**

bounded by the curve $y = \frac{1}{2}\log_e x$, the lines $y = -1$, $y = 2$ and the y-axis

is rotated about the y-axis.

Question 12 continues on the next page.

Question 12 (continued)

(c) Consider the expansion of $\left(1+x\right)^{n-1}$ for integers $n>2$.

(i) Show that: **3**

$$n\binom{n-1}{1}+n\binom{n-1}{2}+\ldots+n\binom{n-1}{n-2}=n\left(2^{n-1}-2\right)$$

(ii) Find the smallest positive integer *n* such that: **1**

$$n\binom{n-1}{1}+n\binom{n-1}{2}+\ldots+n\binom{n-1}{n-2}>5000$$

End of Question 12

Question 13 Begin a new page. **(10 marks)**

(a) A particle is moving in a straight line and is oscillating backwards and forwards. At time t seconds, it has displacement x metres from a fixed point O on the line, where $x = A\cos\left(\frac{\pi}{4}t + \alpha\right)$, $A > 0$, and $0 < \alpha < \frac{\pi}{2}$.

After 1 second the particle is 2 metres to the right of O, and after 3 seconds it is 4 metres to the left of O.

Given that $A\cos\alpha - A\sin\alpha = 2\sqrt{2}$ and $A\cos\alpha + A\sin\alpha = 4\sqrt{2}$:

(i) Solve the equations simultaneously, to find the values of A and α in exact form. **2**

(ii) Show that the particle first passes through O after $\frac{4}{\pi}\tan^{-1}3$ seconds. **2**

Question 13 continues on the next page.

Question 13 (continued)

(b) The position coordinates of any point on the path of a projectile at time $t \geq 0$ in seconds, with initial velocity v ms^{-1} at an angle of projection θ, and acceleration downwards due to gravity, g, are:

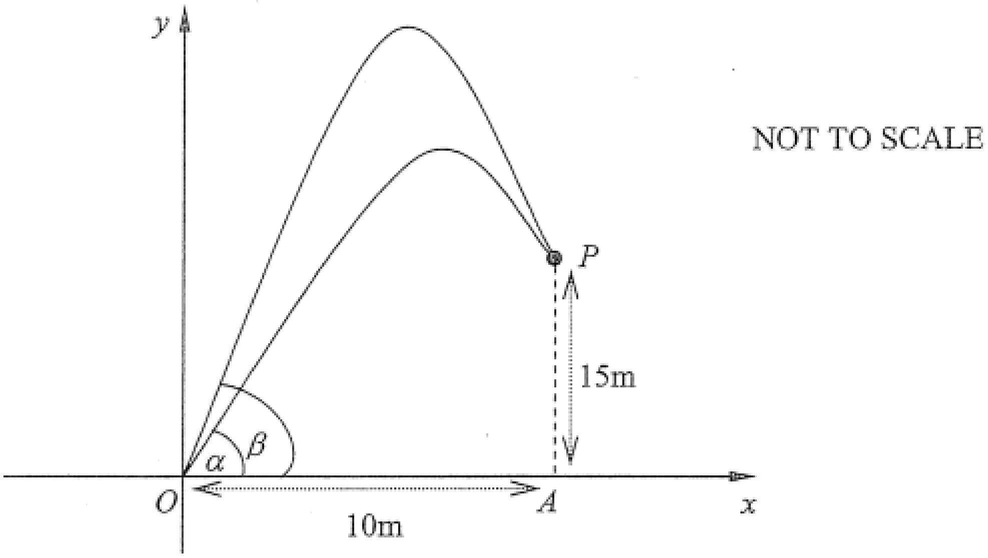
$$x = vt\cos\theta \text{ and } y = vt\sin\theta - \frac{1}{2}gt^2$$

(i) Show that the equation of the path of a projectile is given by: **2**

$$y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta$$

Harry throws a tennis ball from a fixed point O on level ground, with a velocity $v = 7\sqrt{10}$ ms^{-1} at an angle β with the horizontal. Shortly afterwards he throws another tennis ball from the same point at the same speed but at a different angle to the horizontal, α, where $\alpha < \beta$ as shown.

The two tennis balls collide at a point P, vertically above the point A on the ground, where $OA = 10$ m and $AP = 15$ m. The acceleration downwards due to gravity is $g = 9.8$ ms^{-2}.



(ii) Show that $\tan\alpha = 2$ and $\tan\beta = 8$. **2**

(iii) Find, in surd form, the time elapsed between when the tennis balls were thrown. **2**

End of paper

SOLUTIONS

LABOR OMNIA VINCIT

Sydney Girls High School

2021 Alternate Task 4

Mathematics Extension 1

General Instructions

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- Working time – 60 minutes
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Total marks : 40

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2021 Extension 1 Task 4

Multiple choice

1. C
2. A
3. B
4. A
5. D
6. A
7. B
8. C
9. D
10. C

Section I

10 marks
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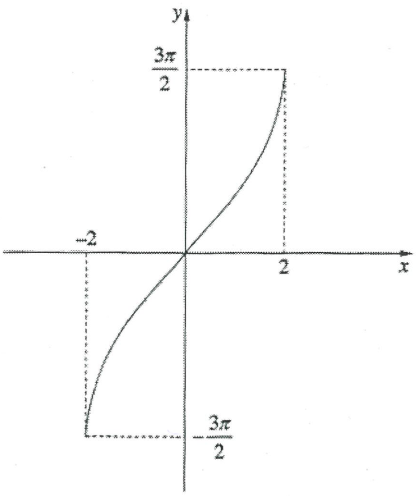
1. Which function best describes the following graph?

(A) $y = 3\sin^{-1} 2x$

(B) $y = \frac{3}{2}\sin^{-1} 2x$

[✓ (C)] $y = 3\sin^{-1}\frac{x}{2}$

(D) $y = \frac{3}{2}\sin^{-1}\frac{x}{2}$



D: $-1 \le \boxed{\frac{x}{2}} \le 1$

$-2 \le x \le 2$.

R: $-\frac{\pi}{2} \le \boxed{\frac{y}{3}} \le \frac{\pi}{2}$

$-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$.

$\therefore \frac{y}{3} = \sin^{-1}\frac{x}{2}$

$y = 3\sin^{-1}\frac{x}{2}$.

2. In the cartesian plane, a vector perpendicular to the line $3x + 2y + 1 = 0$ is

[✓ (A)] $3\underset{\sim}{i} + 2\underset{\sim}{j}$

(B) $-\frac{1}{2}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j}$

(C) $2\underset{\sim}{i} - 3\underset{\sim}{j}$

(D) $\frac{1}{2}\underset{\sim}{i} - \frac{1}{3}\underset{\sim}{j}$

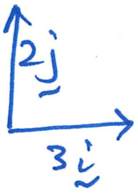
$3x + 2y + 1 = 0$:

$2y = -3x - 1$

$y = -\frac{3}{2}x - \frac{1}{2}$

$m_l = -\frac{3}{2}$

$\therefore m_v = \frac{2}{3}$



$\therefore 3\underset{\sim}{i} + 2\underset{\sim}{j}$

3. Which group of three numbers could be the roots of the polynomial equation

$$x^3 + px^2 - 26x + 24 = 0\ ?$$

$$\overset{}{x^3} \overset{-b}{+px^2} \overset{+c}{- 26x} \overset{-d}{+24} = 0.$$

(A) 2, 3, 4

[✓ (B)] 1, −6, 4

(C) −1, −2, 12

(D) −1, −3, −8

$\alpha\beta\gamma = -24$

$2 \times 3 \times 4 \neq -24$ ∴ not Ⓐ

$-1 \times -2 \times 12 \neq -24$ ∴ not Ⓒ

$\alpha\beta + \beta\gamma + \alpha\gamma = -26$

$1 \times -6 + 1 \times 4 + -6 \times 4 = -6 + 4 - 24$

$= -26$

∴ Ⓑ.

4. $$\frac{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}}{\cot\frac{\theta}{2} + \tan\frac{\theta}{2}} =$$

[✓ (A)] $\cos\theta$

(B) $\sec\theta$

(C) $\tan\theta$

(D) $\cot\theta$

Let $t = \tan\frac{\theta}{2}$

$$\Rightarrow \frac{\frac{1}{t} - t}{\frac{1}{t} + t}$$

$$= \frac{1-t^2}{t} \div \frac{1+t^2}{t}$$

$$= \frac{1-t^2}{t} \times \frac{t}{1+t^2}$$

$$= \frac{1-t^2}{1+t^2} = \cos\theta$$ Ⓐ

5. Which expression is equal to $\int \cos^2 \frac{2x}{5}\, dx$?

(A) $\frac{x}{2} - \frac{2}{5}\sin\frac{4x}{5} + C$

(B) $\frac{x}{2} + \frac{2}{5}\sin\frac{4x}{5} + C$

(C) $\frac{x}{2} - \frac{5}{8}\sin\frac{4x}{5} + C$

[✓ (D)] $\frac{x}{2} + \frac{5}{8}\sin\frac{4x}{5} + C$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.

$\cos^2\left(\frac{2x}{5}\right) = \frac{1}{2} + \frac{1}{2}\cos\left(2 \times \frac{2x}{5}\right)$

$\cos^2\left(\frac{2x}{5}\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{4x}{5}\right)$

$\therefore \int \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{4x}{5}\right)\right) dx$

$= \frac{1}{2}x + \frac{1}{2} \times \frac{5}{4}\sin\frac{4x}{5}$

$= \frac{1}{2}x + \frac{5}{8}\sin\frac{4x}{5} + C$

6. In the expression $(2x + k)^6$ the coefficients of x and x^2 are equal.

What is the value of k?

[✓ (A)] 5

(B) 6

(C) 11

(D) 12

$$(2x+k)^6 = {}^6C_0(2x)^6 + {}^6C_1(2x)^5k + {}^6C_2(2x)^4k^2 + {}^6C_3(2x)^3k^3 + {}^6C_4(2x)^2k^4 + {}^6C_5(2x)k^5 + {}^6C_6(2x)^0k^6.$$

coeff of x = coeff of x^2

$${}^6C_5 \times 2 \times k^5 = {}^6C_4 \times 2^2 \times k^4$$

$${}^6C_5 \times k = {}^6C_4 \times 2$$

$$k = \frac{{}^6C_4 \times 2}{{}^6C_5} = \frac{30}{6} \quad \therefore k = 5.$$

7. A group consisting of two adults, two boys and two girls is to be seated at a round table. The adults are to be seated together. The boys and girls are to sit in alternating positions? How many different seating arrangements are possible?

2 Adults
2 girls
2 boys
6.

(A) 8

[✓ (B)] 16

(C) 24

(D) 30

- Seat the first adult anywhere.
- 2nd adult has 2 ways.
- seat a first boy has 4 ways.
- 2nd boy has only 1 way.
- two girls sit 2 ways

∴ seating arrangements

$= 2 \times 4 \times 2$

$= 16.$

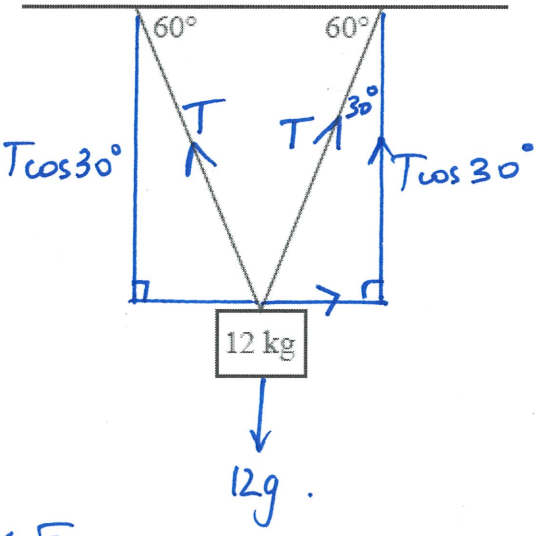
8. A 12kg mass is suspended in equilibrium from a horizontal ceiling by two identical light strings. Each string makes an angle of 60° with the ceiling, as shown. If the force of gravity is g m/s^2 then the magnitude, in newtons, of the tension in each string is equal to:

(A) $12g$

(B) $24g$

[✓ (C) $4\sqrt{3}g$]

(D) $8\sqrt{3}g$



$2T\cos 30^\circ = 12g$

$T = \dfrac{12g}{2\cos 30^\circ}$

$T = \dfrac{12g}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = 4\sqrt{3}g$.

9. The volume V of a spherical balloon of radius r mm is increasing at a constant rate of 800 mm^3 per second. What is the rate of change of the radius with respect to time?

(A) $\dfrac{\pi r}{100}$ mm/second

(B) $\dfrac{100}{\pi r}$ mm/second

(C) $\dfrac{\pi r^2}{200}$ mm/second

[✓ (D) $\dfrac{200}{\pi r^2}$ mm/second]

$\dfrac{dV}{dt} = 800$ mm^3/second . Find $\dfrac{dr}{dt}$.

$V = \dfrac{4}{3}\pi r^3$

$\dfrac{dV}{dr} = 4\pi r^2$.

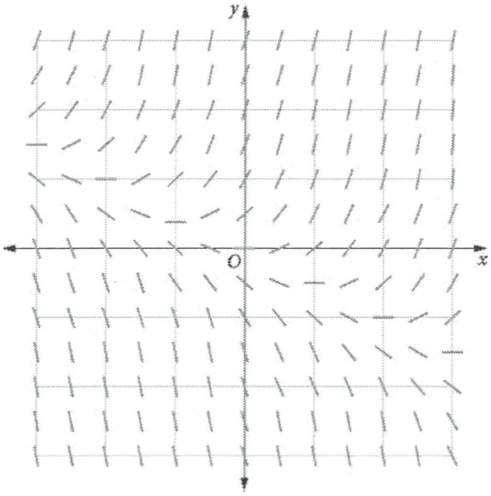
$\therefore \dfrac{dr}{dt} = \dfrac{dV}{dt} \times \dfrac{dr}{dV}$

$= 800 \times \dfrac{1}{4\pi r^2}$

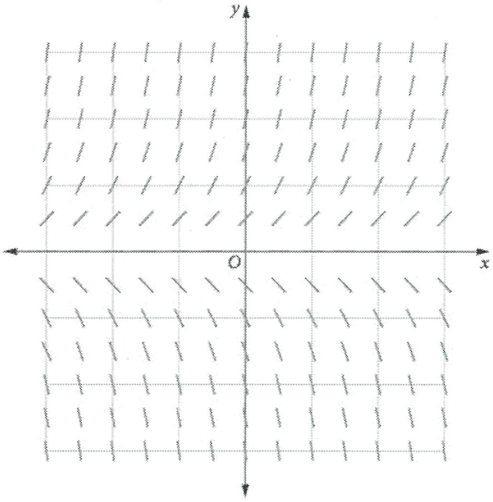
$= \dfrac{200}{\pi r^2}$ mm/s .

10. Which of the following direction fields could have the differential equation $\frac{dy}{dx} = x - ky$ as a solution if $k > 0$?

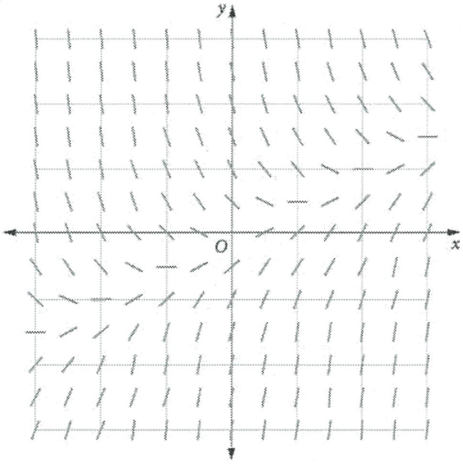
(A)



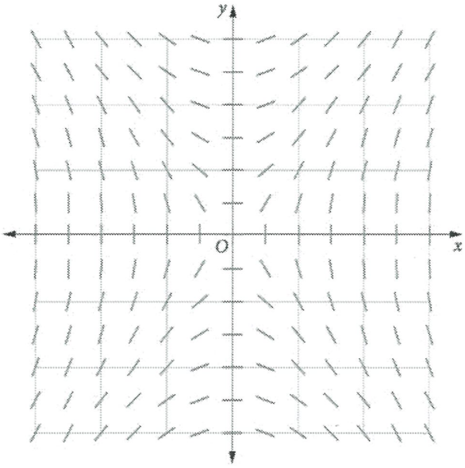
(B)



[✓ (C)]



(D)



when $\frac{dy}{dx} = 0$: $ky = x \Rightarrow y = \frac{x}{k}$ (line through origin, having horizontal line elements.

∴ not Ⓑ and not Ⓓ.

But $k > 0$, ∴ $y = \frac{x}{k}$ ⤢ ∴ Ⓒ.

Question 11 Begin a new Page **(10 marks)**

(a)

(i) Show that $(x-3)$ is a factor of the polynomial **1**

$P(x)=x^3-4x^2+x+6$.

(ii) Hence, express $P(x)$ in factored form. **2**

i) $P(3)=3^3-4(3)^2+3+6$

$\therefore P(3)=0$. ✓ ①

ii)
$$\begin{array}{r|l} & x^2-x-2 \\ \hline x-3 & x^3-4x^2+x+6 \\ & \underline{x^3-3x^2} \\ & \quad -x^2+x \\ & \quad \underline{-x^2+3x} \\ & \qquad -2x+6 \\ & \qquad \underline{-2x+6} \\ & \qquad\qquad 0 \end{array}$$

① for working if answer is incorrect.

$\therefore P(x)=(x-3)(x^2-x-2)$

$\underline{P(x)=(x-3)(x-2)(x+1)}$ ✓✓ ②

(b) Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\sin\frac{x}{3}}{4x}\right)$ **1**

$$\lim_{x\to 0}\left(\frac{\sin\frac{x}{3}}{4x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin\frac{x}{3}}{\frac{x}{3}}\right)\times\frac{\left(\frac{x}{3}\right)}{4x}$$

$$= 1\times\frac{\cancel{x}}{3}\times\frac{1}{4\cancel{x}}$$

$$=\frac{1}{12}$$ ✓ ①

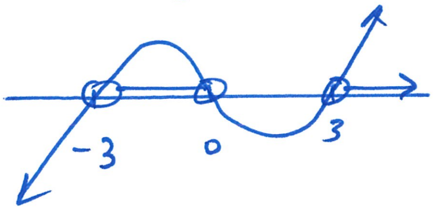
(c) Solve: $\frac{x^2-9}{3x} > 0$ **2**

$$\frac{x^2-9}{3x} > 0$$

① for working if answer is incorrect

$$3x^2 \times \frac{x^2-9}{3x} > 0 \times 3x^2$$

$$x(x-3)(x+3) > 0 .$$



By observation:

$\underline{\underline{-3 < x < 0 \quad \text{and} \quad x > 3}}$. ✓✓ ②.

(d) Evaluate: **2**

$$\int \frac{2x}{\sqrt{25-x^2}}\,dx$$

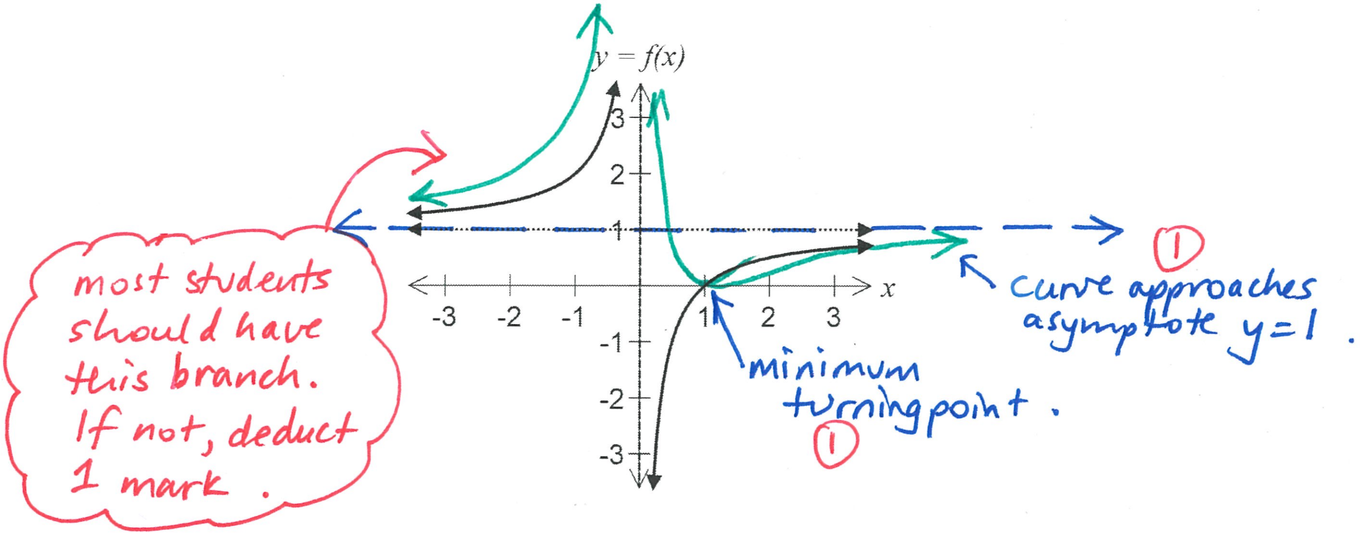
$$\int \frac{2x}{\sqrt{25-x^2}}\,dx .$$

$$= \int 2x.(25-x^2)^{-\frac{1}{2}}\,dx \quad ✓$$

$$= -\frac{1}{\frac{1}{2}}(25-x^2)^{\frac{1}{2}} + c \quad \text{(reverse chain rule)}.$$

$$= \underline{\underline{-2\sqrt{25-x^2} + c}} \quad ✓ \qquad ②.$$

(e) Consider the function $f(x)=\dfrac{x-1}{x}$, which is graphed below:



most students should have this branch. If not, deduct 1 mark.

① curve approaches asymptote y=1.

minimum turning point. ①

Using the graph of $y=f(x)$, sketch the function $y=\{f(x)\}^{2}$. **2**

End of Question 11

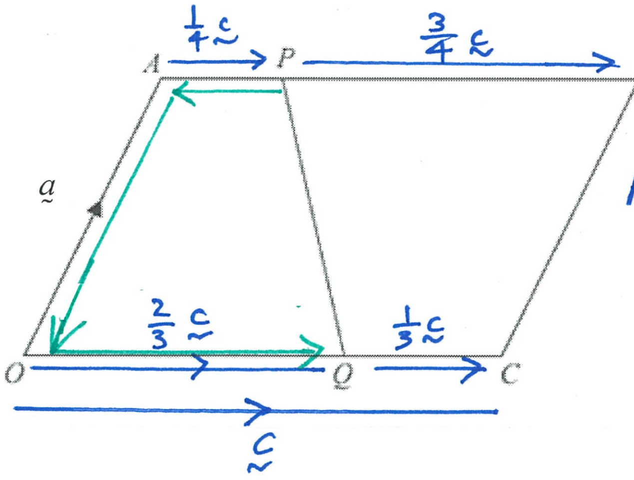
Question 12 Begin a new page. **(10 marks)**

(a) In the diagram below, $OABC$ is a parallelogram. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$ **3**

P is the point on AB such that $AP = \frac{1}{4}AB$.

Q is the point on OC such that $OQ = \frac{2}{3}OC$.

Find $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$, giving your answer in simplest form.



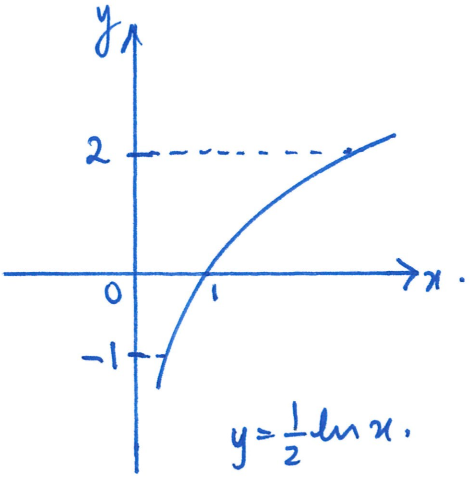
$$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AO} + \overrightarrow{OQ} \quad ✓$$

$$= -\frac{1}{4}\underset{\sim}{c} - \underset{\sim}{a} + \frac{2}{3}\underset{\sim}{c} \quad ✓$$

$$= \left(\frac{2}{3} - \frac{1}{4}\right)\underset{\sim}{c} - \underset{\sim}{a}$$

$$= \frac{5}{12}\underset{\sim}{c} - \underset{\sim}{a} \quad ✓ \quad ③$$

(b) Find the volume, in terms of π, of the solid formed when the area bounded by the curve $y = \frac{1}{2}\log_e x$, the lines $y = -1$, $y = 2$ and the y-axis is rotated about the y-axis. **3**



$$y = \frac{1}{2}\ln x$$

$$2y = \ln x$$

$$e^{2y} = x \quad ✓$$

$$V = \pi \int_a^b x^2 \, dy$$

$$= \pi \int_{-1}^{2} (e^{2y})^2 \, dy \quad ✓$$

$$= \pi \int_{-1}^{2} e^{4y} \, dy$$

$$= \left[\frac{e^{4y}}{4}\right]_{-1}^{2} \times \pi$$

$$V = \left(\frac{e^8}{4} - \frac{e^{-4}}{4}\right)\pi \text{ units}^3 \quad ✓ \quad ③$$

Question 12 (continued)

(c) Consider the expansion of $(1+x)^{n-1}$ for integers $n>2$.

(i) Show that: 3

$$n\binom{n-1}{1}+n\binom{n-1}{2}+\ldots+n\binom{n-1}{n-2}=n\left(2^{n-1}-2\right)$$

(ii) Find the smallest positive integer n such that: 1

$$n\binom{n-1}{1}+n\binom{n-1}{2}+\ldots+n\binom{n-1}{n-2}>5000$$

i) $(1+x)^{n-1} = 1 + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \dots + \binom{n-1}{n-2}x^{n-2} + x^{n-1}$

Let $x=1$:

$(1+1)^{n-1} = 1 + \binom{n-1}{1} + \binom{n-1}{2} + \dots + \binom{n-1}{n-2} + 1$ ✓

$2^{n-1} = 2 + \binom{n-1}{1} + \binom{n-1}{2} + \dots + \binom{n-1}{n-2}$

$2^{n-1}-2 = \binom{n-1}{1} + \binom{n-1}{2} + \dots + \binom{n-1}{n-2}$ ✓

$n(2^{n-1}-2) = n\left[\binom{n-1}{1} + \binom{n-1}{2} + \dots + \binom{n-1}{n-2}\right]$ ✓ ③

$\therefore n\binom{n-1}{1} + n\binom{n-1}{2} + n\binom{n-1}{3} + \dots + n\binom{n-1}{n-2} = n(2^{n-1}-2).$

ii) Solve $n(2^{n-1}-2) > 5000$

Use trial and error:

$n=8$: $8(2^7-2)=1008$

$n=9$: $9(2^8-2)=2286$

$n=10$: $10(2^9-2)=5100$ $\therefore$ $n=10$ ✓ ①

Question 13 Begin a new page. **(10 marks)**

(a) A particle is moving in a straight line and is oscillating backwards and forwards. At time t seconds, it has displacement x metres from a fixed point O on the line, where $x = A\cos\left(\frac{\pi}{4}t + \alpha\right)$, $A > 0$, and $0 < \alpha < \frac{\pi}{2}$.

After 1 second the particle is 2 metres to the right of O, and after 3 seconds it is 4 metres to the left of O.

Given that $A\cos\alpha - A\sin\alpha = 2\sqrt{2}$ and $A\cos\alpha + A\sin\alpha = 4\sqrt{2}$:

(i) Solve the equations simultaneously, to find the values of A and α in exact form. **2**

(ii) Show that the particle first passes through O after $\frac{4}{\pi}\tan^{-1}3$ seconds. **2**

i) $A\cos\alpha - A\sin\alpha = 2\sqrt{2}$ ... ①

$A\cos\alpha + A\sin\alpha = 4\sqrt{2}$ ... ②

①+② : $2A\cos\alpha = 6\sqrt{2} \Rightarrow A\cos\alpha = 3\sqrt{2}$

② −① : $2A\sin\alpha = 2\sqrt{2} \Rightarrow A\sin\alpha = \sqrt{2}$

$\therefore \quad A^2(\cos^2\alpha + \sin^2\alpha) = (3\sqrt{2})^2 + (\sqrt{2})^2$

$A^2 = 20$

$\underline{A = 2\sqrt{5}, \ A > 0.}$ ✓

$\frac{A\sin\alpha}{A\cos\alpha} = \frac{\sqrt{2}}{3\sqrt{2}}$

$\therefore \tan\alpha = \frac{1}{3}$

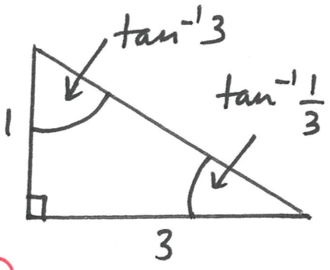
✓

$\underline{\alpha = \tan^{-1}\frac{1}{3}} \quad \left(0 < \alpha < \frac{\pi}{2}\right)$ ②.

ii) when $x=0$: $\cos\left(\frac{\pi}{4}t+\alpha\right)=0$

$t>0$: $\frac{\pi}{4}t+\alpha=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

First such $t>0$: $\frac{\pi t}{4}=\frac{\pi}{2}-\alpha$ ①



$\frac{\pi}{4}t=\frac{\pi}{2}-\tan^{-1}\frac{1}{3}$

$\frac{\pi}{4}t=\tan^{-1}3$ ①

$\therefore \quad \underline{\underline{t=\frac{4}{\pi}\tan^{-1}3}}\,.$

(b) The position coordinates of any point on the path of a projectile at time $t \geq 0$ in seconds, with initial velocity v ms^{-1} at an angle of projection θ, and acceleration downwards due to gravity, *g*, are:

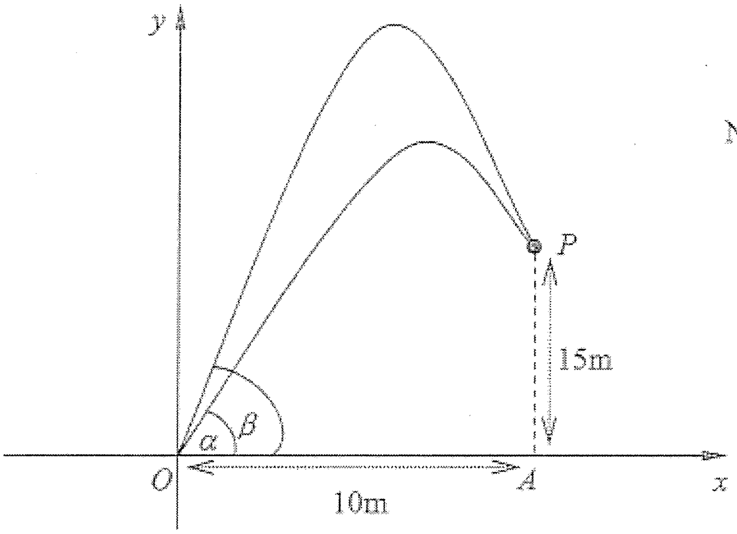
$$x = vt\cos\theta \text{ and } y = vt\sin\theta - \frac{1}{2}gt^2$$

(i) Show that the equation of the path of a projectile is given by: **2**

$$y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta$$

Harry throws a tennis ball from a fixed point *O* on level ground, with a velocity $v = 7\sqrt{10}$ ms^{-1} at an angle β with the horizontal. Shortly afterwards he throws another tennis ball from the same point at the same speed but at a different angle to the horizontal, α, where $\alpha < \beta$ as shown.

The two tennis balls collide at a point *P*, vertically above the point *A* on the ground, where $OA = 10$ m and $AP = 15$ m. The acceleration downwards due to gravity is $g = 9.8$ ms^{-2}.



NOT TO SCALE

(ii) Show that $\tan\alpha = 2$ and $\tan\beta = 8$. **2**

(iii) Find, in surd form, the time elapsed between when the tennis balls were thrown. **2**

End of paper

b) i) $x = vt\cos\theta$ and $y = vt\sin\theta - \frac{1}{2}gt^2$

$\Rightarrow t = \dfrac{x}{v\cos\theta}$ sub into y:

$$y = (v\sin\theta) \cdot \frac{x}{v\cos\theta} - \frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2$$ ✓

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$$

$$\underline{\underline{y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta}}.$$ ✓ ②

ii) At P: $x=10$, $y=15$ and $g=9.8$, $v = 7\sqrt{10}$.

Using i): $15 = 10\tan\theta - \dfrac{9.8(10)^2}{2(7\sqrt{10})^2}\sec^2\theta$

$$15 = 10\tan\theta - \frac{9.8 \times 100}{2 \times 49 \times 10}(1+\tan^2\theta)$$ ✓

some progress

$$15 = 10\tan\theta - 1 - \tan^2\theta$$

$$0 = \tan^2\theta - 10\tan\theta + 16$$

$$0 = (\tan\theta - 8)(\tan\theta - 2)$$

$\therefore \tan\theta = 8$ or $\tan\theta = 2$ ✓

since $\alpha < \beta$: $\tan\beta = 8$ and $\tan\alpha = 2$. ②

iii) Consider the two paths and find the time travelled to reach P:

Tennis ball 1: $v=7\sqrt{10}$, $\theta=\beta$, $\tan\beta=8$ and $x=10$:

$$t_1 = \frac{10}{7\sqrt{10}\cos\beta}$$

$$= \frac{10\sec\beta}{7\sqrt{10}} \qquad \left(\begin{array}{l}\sec^2\beta = 1+\tan^2\beta \\ \sec^2\beta = 1+8^2\end{array}\right)$$

$$= \frac{10\times\sqrt{65}}{7\sqrt{10}}$$

$$t_1 = \frac{\sqrt{650}}{7}$$

✓ ① progress towards a solution.

Tennis ball 2: $v=7\sqrt{10}$, $\theta=\alpha$, $\tan\alpha=2$ and $x=10$:

$$t_2 = \frac{10}{7\sqrt{10}\cos\alpha}$$

$$= \frac{10\sec\alpha}{7\sqrt{10}} \qquad \left(\begin{array}{l}\sec^2\alpha = 1+\tan^2\alpha \\ \sec^2\alpha = 1+4\end{array}\right)$$

$$= \frac{10\sqrt{5}}{7\sqrt{10}}$$

$$= \frac{\sqrt{50}}{7}$$

∴ time elapsed: $t_1 - t_2 = \dfrac{\sqrt{650}-\sqrt{50}}{7}$ seconds.

✓ ②